

ON PRIME VALUES OF CYCLOTOMIC POLYNOMIALS

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ABSTRACT. We present several approaches on finding necessary and sufficient conditions on n so that $\Phi_k(x^n)$ is irreducible where Φ_k is the k -th cyclotomic polynomial.

1. INTRODUCTION

Let $U_n = \{z \in \mathbf{C} \mid z^n = 1\}$ denote the set of n th roots of unity. U_n is a group under complex multiplication, in fact a cyclic group. A complex number ω is called a *primitive n th root of unity* provided ω is an n th root of unity and has order n in the group U_n . Such an ω generates the group, i.e. $\omega, \omega^2, \dots, \omega^n = 1$ coincides with the set U_n . An element of U_n of the form ω^k is a primitive root of unity iff k is relatively prime to n . Therefore the number of primitive roots of unity is equal to $\phi(n)$ where ϕ is Euler's totient function.

There are three cube roots of unity: $1, \omega = (-1 + \sqrt{3}i)/2$ and ω^2 ; only ω and ω^2 are primitive. There are two primitive fourth roots of unity: $\pm i$. They are roots of the polynomial $x^2 + 1$.

We define the n th cyclotomic polynomial by

$$\Phi_n(x) = (x - \omega_1)(x - \omega_2) \cdots (x - \omega_s),$$

where $\omega_1, \omega_2, \dots, \omega_s$ are all the distinct primitive n th roots of unity. The degree of Φ_n is of course equal to $s = \phi(n)$. $\Phi_n(x)$ is a monic, irreducible polynomial with integer coefficients. The first twenty cyclotomic polynomials are given below:

$$\begin{aligned} \Phi_1(x) &= x - 1 \\ \Phi_2(x) &= x + 1 \\ \Phi_3(x) &= x^2 + x + 1 \\ \Phi_4(x) &= x^2 + 1 = \Phi_2(x^2) \\ \Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\ \Phi_6(x) &= x^2 - x + 1 = \Phi_3(-x) \\ \Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\ \Phi_8(x) &= x^4 + 1 = \Phi_2(x^4) = \Phi_4(x^2) \\ \Phi_9(x) &= x^6 + x^3 + 1 = \Phi_3(x^3) \\ \Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 = \Phi_5(-x) \\ \Phi_{11}(x) &= x^{10} + x^9 + \cdots + x + 1 \\ \Phi_{12}(x) &= x^4 - x^2 + 1 = \Phi_6(x^2) \\ \Phi_{13}(x) &= x^{12} + x^{11} + \cdots + x + 1 \\ \Phi_{14}(x) &= x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = \Phi_7(-x) \\ \Phi_{15}(x) &= x^8 - x^7 + x^5 - x^4 + x^3 - x + 1 = \frac{\Phi_3(x^5)}{\Phi_3(x)} = \frac{\Phi_5(x^3)}{\Phi_5(x)} \\ \Phi_{16}(x) &= x^8 + 1 = \Phi_2(x^8) = \Phi_8(x^2) \\ \Phi_{17}(x) &= x^{16} + x^{15} + \cdots + x + 1 \\ \Phi_{18}(x) &= x^6 - x^3 + 1 = \Phi_9(-x) \\ \Phi_{19}(x) &= x^{18} + x^{17} + \cdots + x + 1 \\ \Phi_{20}(x) &= x^8 - x^6 + x^4 - x^2 + 1 = \Phi_{10}(x^2) \end{aligned}$$

A basic formula for cyclotomic polynomials is

$$(1) \quad x^n - 1 = \prod_{d|n} \Phi_d(x) ,$$

where d ranges over all positive divisors of n . This gives a recursive method of calculating cyclotomic polynomials. For example, if $n = p$, where p is prime, then $x^p - 1 = \Phi_1(x)\Phi_p(x)$ which implies that

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 .$$

Using Möbius inversion on (1) we obtain an explicit formula for cyclotomic polynomials in terms of the Möbius function:

$$(2) \quad \Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(\frac{n}{d})} .$$

We record the following formulas where p is a prime and n, m are positive integers. The formulas can be found in many elementary textbooks.

If $p|m$ then

$$(3) \quad \Phi_{pm}(x) = \Phi_m(x^p)$$

If $p \nmid m$

$$(4) \quad \Phi_{pm}(x)\Phi_m(x) = \Phi_m(x^p)$$

$$(5) \quad \Phi_{p^k}(x) = \Phi_p(x^{p^{k-1}})$$

If n is odd, $n > 1$

$$(6) \quad \Phi_{2n}(x) = \Phi_n(-x)$$

Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and define $m = p_1 p_2 \cdots p_s$. Then

$$(7) \quad \Phi_n(x) = \Phi_m(x^{\frac{n}{m}})$$

In this paper we deal with the following question:

Question 1. Find conditions on k and n so that $\Phi_k(x^n)$ is irreducible. This gives a necessary condition for $\Phi_k(a^n)$ to be a prime if $a, n, k \in \mathbf{Z}^+$, $k > 1$, $a > 1$.

A more difficult question which we do not address here is the following. For fixed k and n (or fixed k and a) are there infinitely many primes of the form $\Phi_k(a^n)$? e.g. If $n = 1$, are there infinitely many primes of the form $\Phi_k(a)$ $a = 1, 2, 3, \dots$ where Φ_k is a given cyclotomic polynomial? Another possibility is to fix a and n . For example primes of the form $\Phi_k(1)$, $\Phi_k(2)$ are investigated in [4].

Example 1 ($k=4$). Let us consider the case $k = 4$, i.e. $\Phi_4(x) = x^2 + 1$. We can rephrase the question as follows: When is $x^{2n} + 1$ irreducible? It is elementary to see that n must be of the form 2^j for some $j \geq 0$. Therefore the polynomial should be of the form

$$x^{2^n} + 1 .$$

In order for $a^{2^n} + 1$ to be prime, it is easy to see that a should be even. In the case $a = 2$, we end-up with the Fermat primes of the form

$$F_n = 2^{2^n} + 1 .$$

The only known Fermat primes are F_0, F_1, F_2, F_3 and F_4 .

Example 2 ($k=1,2$). For completeness let us consider also the cases $k = 1$ and $k = 2$. In the case $k = 1$ we have $\Phi_1(x^n) = x^n - 1$ which is always reducible, unless $n = 1$. Numbers of the form $a - 1$ are prime for infinitely many values of a as was observed by Euclid. In connection with Question 1 we make the following remark: If $\Phi_1(a^n) = a^n - 1$ is prime then $n = p$ for some prime p and $a = 2$. Therefore for Mersenne primes of the form $2^p - 1$ it turns out that the corresponding polynomial $x^p - 1$ is reducible. But this is an exceptional case (That is why we assume $k > 1$ in Question 1). The only other exception is when $a = 1$ and k has at least two distinct prime divisors.

In the case $k = 2$ we have $\Phi_2(x^n) = x^n + 1$. If n is odd, then $\Phi_2(x^n)$ is divisible by $x + 1$. Therefore n must be even and we end-up with a polynomial of the form $x^{2n} + 1$, a case that has been already considered.

Most of the results of this paper are well-known. But to our knowledge they are scattered in various places and the problem in question has not been addressed before. In the next two sections we consider in detail the case $k = 3$. Although the result is stated and proved in two different ways for this particular case, the method of proof is applicable for other values of k . In section 4 we give a third proof which covers the general case of arbitrary k . In section 5 we outline two other proofs of theorem 2, one due to S. Golomb, and one due to L. Leroux.

2. THE CASE $k = 3$

In this section we use the notation

$$p_n(x) = \Phi_3(x^n) = x^{2n} + x^n + 1.$$

It is easy to see that $p_n(x)$ is a product of cyclotomic polynomials. We note that $p_n(x)$ is a monic polynomial, with integer coefficients, having all of its roots in the unit disc (in fact on the unit circle). It follows from a classical result of Kronecker, see [2], that $p_n(x)$ is a product of cyclotomic polynomials. Showing that the roots of this polynomial are roots of unity is easy. Let ρ be such a root, i.e. $\rho^{2n} + \rho^n + 1 = 0$. We let $\lambda = \rho^n$ and thus $\lambda^2 + \lambda + 1 = 0$. In other words λ is a primitive cubic root of unity. Therefore $\rho^{3n} = 1$, i.e., ρ is a $3n$ root of unity. This fact follows even easier from the identity

$$x^{3n} - 1 = (x^n - 1)(x^{2n} + x^n + 1).$$

In this section we will prove the following result:

Theorem 1. *If $a^{2n} + a^n + 1$ is prime then $n = 3^j$ for some $j = 0, 1, 2, \dots$*

Actually, we will show that $x^{2n} + x^n + 1$ is irreducible iff $n = 3^j$ for some $j = 0, 1, 2, \dots$

Let us see some examples for small values of n

Example 3.

- $n = 1$.

$$p_1(x) = x^2 + x + 1 = \Phi_3(x)$$

which is irreducible.

- $n = 2$

$$p_2(x) = x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1) = \Phi_3(x)\Phi_6(x).$$

- $n = 3$

$$p_3(x) = x^6 + x^3 + 1 = \Phi_9(x)$$

which is irreducible.

- $n = 4$

$$p_4(x) = x^8 + x^4 + 1 = \Phi_3(x)\Phi_6(x)\Phi_{12}(x) .$$

- $n = 5$

$$p_5(x) = x^{10} + x^5 + 1 = \Phi_3(x)\Phi_{15}(x) .$$

- $n = 6$

$$p_6(x) = x^{12} + x^6 + 1 = \Phi_9(x)\Phi_{18}(x) .$$

One may wonder about the cyclotomic polynomials that appear in the factorization of $p_n(x)$. We will shortly give a formula that precisely determines the cyclotomic factors of $x^{2n} + x^n + 1$.

Let us examine briefly the behavior of $x^{2n} + x^n + 1$ for fixed n . There is strong experimental evidence that $a^2 + a + 1$ is prime for infinitely many values of a , i.e.

$a = 1, 2, 3, 5, 6, 8, 12, 14, 15, 17, 20, 21, 24, 27, 33, 38, 41, 50, 54, 57, 59, 62, 66, 69, 71, 75, 77, 78, 80, 89, 90, 99, \dots$. There are a total of 189 such values under 1000.

Similarly, we note that $a^6 + a^3 + 1$ is likely to be prime for infinitely many values of a , i.e.

$$a = 1, 2, 3, 8, 11, 20, 21, 26, 30, 50, 51, 56, 60, 78, 98, \dots$$

There are a total of 79 such values under 1000.

Finally, $a^{18} + a^9 + 1$ is prime for

$$a = 1, 2, 11, 44, 45, 56, 62, 63, 110, 170, 219, 234, 245, 261, 263, \dots$$

There are a total of 47 values under 1000.

On the other hand, we may fix x and investigate the behavior of this expression as a function of n . This will be the analogue of Fermat primes. For example for $x = 2$ we may ask whether

$$2^{23^n} + 2^{3^n} + 1$$

is prime for infinitely many values of n . It is certainly true for $n = 0, 1, 2$. Similarly we may ask whether

$$3^{23^n} + 3^{3^n} + 1$$

is prime for certain values of n . It is true for $n = 0, 1$.

We now prove the following lemma, see [7]:

Lemma 1. *Let $f, g \in \mathbf{Z}[x]$ and denote by $c(f)$ the content of f . If $c(g)|c(f)$ and $g(n)|f(n)$ for infinitely many integers n , then $g|f$ in $\mathbf{Z}[x]$.*

Proof. Divide f by g in $\mathbf{Q}[x]$ to obtain

$$f(x) = g(x)q(x) + r(x)$$

with $q(x), r(x)$ in $\mathbf{Q}[x]$ and $\deg r(x) < \deg g(x)$.

Let m be the least common multiple of the denominators of q and r so that mq and mr are in $\mathbf{Z}[x]$. Let n_i be an infinite sequence such that $g(n_i)|f(n_i)$ for all i . Then

$$m \frac{f(n_i)}{g(n_i)} - mq(n_i) = m \frac{r(n_i)}{g(n_i)} .$$

Note that the left hand side is always an integer. Therefore the right hand side is also an integer for all i . But since the degree of r is strictly less than the degree of g we have

$$\lim_{i \rightarrow \infty} \frac{mr(n_i)}{g(n_i)} = 0 .$$

This implies that $r(n_i) = 0$ for big i . As a result $r(x)$ must be identically zero. We conclude that

$$f(x) = g(x)q(x)$$

with $q(x) \in \mathbf{Q}[x]$. Now we use Gauss Lemma:

$$mc(f) = c(g)c(mq) ,$$

$$m \frac{c(f)}{c(g)} = c(mq) .$$

The left hand side is an integer by assumption and a multiple of m . This shows that all the coefficients of $mq(x)$ are a multiples of m . Therefore, $q(x) \in \mathbf{Z}[x]$. \square

The following lemma will be used in the proof of Theorem (1).

Lemma 2.

$$\Phi_k(x) | \Phi_k(x^n)$$

$$\text{iff } (n, k) = 1.$$

Proof. We prove the following more general result: Let $k \geq 2$. The only polynomials with coefficients in \mathbf{Z} , monic and irreducible which satisfy $p(n) | p(n^k)$ for infinite values of n are

- (1) 1
- (2) x
- (3) Φ_j with $(j, k) = 1$.

Obviously, 1 and x satisfy the conditions of the Theorem and we consider them as trivial solutions. If p is any monic polynomial satisfying $p(n) | p(n^k)$ for infinite values of n then

$$p(x^k) = p(x)Q(x)$$

for some polynomial $Q(x) \in \mathbf{Z}[x]$ (lemma 1). Let ζ be a root of p . It follows that $p(\zeta^k) = p(\zeta)Q(\zeta) = 0$ and therefore ζ^k is also a root. Repeating the argument we see that

$$\zeta, \zeta^k, \zeta^{k^2}, \dots$$

are all roots of $p(x)$. But $p(x)$ has only a finite number of roots and therefore two powers of ζ should be equal. Since $\zeta \neq 0$ (otherwise p would be reducible) it follows that ζ is a root of unity. If ζ is primitive of order j then all the primitive roots of order j are also roots of p . Therefore p is a multiple of Φ_j and being irreducible forces $p = \Phi_j$. In order for ζ^k to be primitive of order j we must have $(k, j) = 1$. \square

As an immediate application of the Lemma we note that

$$\Phi_3(x) | \Phi_3(x^n)$$

in case $(3, n) = 1$. Therefore the polynomial $p_n(x)$ is reducible in case n is relatively prime to 3.

As another application of this result we mention the following "millennial Polynomial problem" [1]: Find all monic, irreducible polynomials of degree 2000 with integer coefficients such that $p(n) | p(n^2)$ for every natural number n . The only solutions are Φ_j , where $j = 2525, 3333, 3765, 4125$.

We now proceed to the proof of theorem (1).

Proof. Suppose $n = 3^{j-1}m$ where $(3, m) = 1$ with $m > 1, j > 1$. First note that

$$\Phi_{3^j}(x) = \Phi_3(x^{3^{j-1}}) = x^{2 \cdot 3^{j-1}} + x^{3^{j-1}} + 1 .$$

Therefore

$$\Phi_{3^j}(x^m) = x^{2m \cdot 3^{j-1}} + x^{m \cdot 3^{j-1}} + 1 = \Phi_3(x^n) .$$

Since $(3^j, m) = 1$, it follows that

$$\Phi_{3^j}(x^m) = \Phi_3(x^n)$$

is reducible by lemma 2.

If $n = 3^j$ then

$$p_n(x) = x^{23^j} + x^{3^j} + 1 = \Phi_{3^{j+1}}(x)$$

which is irreducible. \square

3. ANOTHER APPROACH

In this section we give a second proof of theorem 1 and in addition we determine the factorization of $p_n(x)$. We are still in the case $k = 3$. We have

$$x^{3n} - 1 = \prod_{d|3n} \Phi_d(x) ,$$

and

$$x^n - 1 = \prod_{j|n} \Phi_j(x) .$$

Since

$$x^{3n} - 1 = (x^n - 1)(x^{2n} + x^n + 1) = (x^n - 1)p_n(x)$$

we obtain

$$(8) \quad p_n(x) = \frac{\prod_{d|3n} \Phi_d(x)}{\prod_{j|n} \Phi_j(x)} .$$

This formula can be used to determine the factorization of p_n for specific values of n .

Example 4. Take $n = 12$.

$$p_{12}(x) = \frac{\prod_{d|36} \Phi_d(x)}{\prod_{j|12} \Phi_j(x)} = \Phi_9(x)\Phi_{18}(x)\Phi_{36}(x) .$$

We deduce from formula (8) that the polynomial $p_n(x)$ is reducible only if the difference between the number of divisors of $3n$ from the number of divisors of n is strictly bigger than 1. If we denote by $\tau(n)$ the number of positive divisors of n then the condition is simply $\tau(3n) - \tau(n) > 1$.

We can now give a second proof of theorem (1).

Proof. Suppose $n = 3^{j-1}m$ where $(3, m) = 1$ with $m > 1, j > 1$. Then

$$\tau(3n) - \tau(n) = \tau(3^j m) - \tau(3^{j-1} m) = (j+1)\tau(m) - j\tau(m) = \tau(m) > 1 .$$

\square

Note that this line of proof generalizes easily for the case where $k = p$ a prime number. In such a case,

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 .$$

$$p_n(x) = \Phi_p(x^n) = x^{n(p-1)} + x^{n(p-2)} + \cdots + x^n + 1 .$$

As before we have

$$x^{pn} - 1 = (x^n - 1)p_n(x) .$$

Finally

$$(9) \quad p_n(x) = \frac{\prod_{d|pn} \Phi_d(x)}{\prod_{j|n} \Phi_j(x)} .$$

Suppose $n = p^{j-1}m$ where $(p, m) = 1$ with $m > 1$, $j > 1$. Then

$$\tau(pn) - \tau(n) = \tau(p^j m) - \tau(p^{j-1} m) = (j+1)\tau(m) - j\tau(m) = \tau(m) > 1$$

The conclusion: $p_n(x)$ is irreducible iff $n = p^j$ for $j = 0, 1, 2, \dots$

4. THE GENERAL CASE

We now consider the most general case. We find conditions on k and n so that $\Phi_k(x^n)$ is irreducible. Suppose $k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$, with $\alpha_i > 0$. We claim the following:

Theorem 2. $\Phi_k(x^n)$ is irreducible iff $n = p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}$ for some $\beta_j \geq 0$.

Proof. The proof is based on equation (4) which we noted in the introduction: If p does not divide k then

$$\Phi_{pk}(x)\Phi_k(x) = \Phi_k(x^p) .$$

For completeness we give a proof of this property.

$$\begin{aligned} \Phi_{pk}(x) &= \prod_{d|pk} (x^d - 1)^{\mu(\frac{pk}{d})} \prod_{d|pk} (x^d - 1)^{\mu(\frac{p}{d})} = \\ &= \Phi_k(x^p) \prod_{d|pk} (x^d - 1)^{\mu(\frac{pk}{d})} . \end{aligned}$$

But $\mu\left(\frac{pk}{d}\right) = \mu(p)\mu\left(\frac{k}{d}\right) = -\mu\left(\frac{k}{d}\right)$ and the result follows.

Suppose p is a prime divisor of n such that p does not divide k . Then $n = pt$ for some integer t .

$$\Phi_k(x^n) = \Phi_k(x^{pt}) = \Phi_k((x^t)^p) = \Phi_{pk}(x^t)\Phi_k(x^t) .$$

This implies that $\Phi_k(x^n)$ is reducible. On the other hand if $n = p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}$ then

$$(10) \quad \Phi_k(x^n) = \Phi_k\left(x^{p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}}\right) = \Phi_k\left(x^{\frac{\lambda}{k}}\right) = \Phi_\lambda(x) ,$$

where $\lambda = \prod_{i=1}^s p_i^{\alpha_i + \beta_i} = kn$. Therefore, in this case $\Phi_k(x^n)$ is irreducible. \square

Note that if we set $n = p$, where p is prime, in equation (10) we obtain (3).

Example 5. Take $k = 6$. We have $\Phi_6(x) = x^2 - x + 1$. As usual we denote by $p_n(x) = \Phi_6(x^n) = x^{2n} - x^n + 1$. According to theorem (2), $p_n(x)$ is irreducible iff $n = 2^\alpha 3^\beta$ for some $\alpha, \beta \geq 0$. Indeed

- $n = 1$.

$$p_1(x) = x^2 - x + 1 = \Phi_6(x)$$

- $n = 2$

$$p_2(x) = x^4 - x^2 + 1 = \Phi_6(x^2) = \Phi_{6 \cdot 2}(x) = \Phi_{12}(x)$$

- $n = 3$

$$p_3(x) = x^6 - x^3 + 1 = \Phi_6(x^3) = \Phi_{18}(x) .$$

- $n = 4$

$$p_4(x) = x^8 - x^4 + 1 = \Phi_6(x^4) = \Phi_6(x^{\frac{24}{6}}) = \Phi_{24}(x)$$

- $n = 5$

$$p_5(x) = x^{10} - x^5 + 1 = \Phi_6(x)\Phi_{30}(x) .$$

- $n = 6$

$$p_6(x) = x^{12} - x^6 + 1 = \Phi_6(x^{\frac{36}{6}}) = \Phi_{36}(x) .$$

- $n = 7$

$$p_7(x) = x^{14} - x^7 + 1 = \Phi_6(x)\Phi_{42}(x) .$$

Note that in this example the factorization is determined by the formula:

$$p_n(x) = \frac{\prod_{d|6n} \Phi_d(x) \prod_{s|n} \Phi_s(x)}{\prod_{j|2n} \Phi_j(x) \prod_{t|3n} \Phi_t(x)} .$$

5. RELATED WORKS

After the posting of preprint [3] we became aware of the following two alternative approaches to theorem (2). The first is due to Solomon Golomb dating back to 1978. The proof in [6] is the following:

Since the roots of $\Phi_k(x)$ are the primitive k th roots of unity, the roots of $\Phi_k(x^n)$ are the n th roots of these, which are the (nk) th roots of unity, including all the primitive (nk) th roots of unity. Thus $\Phi_{nk}(x)$ divides $\Phi_k(x^n)$, and $\Phi_k(x^n)$ can only be irreducible if it is equal to $\Phi_{nk}(x)$. On the other hand, since all cyclotomic polynomials are irreducible over the rationals, $\Phi_k(x^n)$ is irreducible if and only if it equals $\Phi_{nk}(x)$. Since $\Phi_{nk}(x)$ divides $\Phi_k(x^n)$, these two polynomials are equal if and only if they have the same degree. Therefore we must have $\phi(kn) = k\phi(n)$. This condition is equivalent to

$$\prod_{p|nk} \left(1 - \frac{1}{p}\right) = \prod_{p|k} \left(1 - \frac{1}{p}\right) .$$

Clearly this condition holds iff n has no prime factors not contained in k .

There is also a proof along the same lines by Louis Leroux in [5] which also includes a nice formula for factoring $\Phi_k(x^n)$. Recall that if $n = \prod_i p_i^{\alpha_i}$ then $\alpha_i = \text{ord}_{p_i}(n)$. Let

$$m = \prod_{p|k} p^{\text{ord}_p(n)}$$

and $N = \frac{n}{m}$. Then

$$(11) \quad \Phi_k(x^n) = \Phi_{km}(x^N) = \prod_{d|N} \Phi_{kmd}(x) .$$

To prove the first equality it is enough to remark that the two polynomials have the same roots. Since these polynomials are monic, square free and of the same degree they must be equal. Similarly, each root of the first polynomial is a root of the third. These two polynomials also have the same degree due to the fact

$$\sum_{d|N} \phi(d) = N .$$

If each prime in n is also in k then $m = n$ and $N = 1$. Therefore using (11) we conclude that $\Phi_k(x^n) = \Phi_{kn}(x)$ which is irreducible. Also note that the number of factors of $\Phi_k(x^n)$ is $\tau(N)$. Let us repeat example (4) using formula (11).

Example 6. We have $k = 3$ and $n = 12$. The order of 3 in 12 is 1 and therefore $m = 3$, $N = 4$. We obtain

$$\Phi_3(x^{12}) = \prod_{d|4} \Phi_{9d}(x) = \Phi_9(x)\Phi_{18}(x)\Phi_{36}(x) .$$

REFERENCES

- [1] D. Caragea, V. Ene, Problem 10802, *The American Mathematical Monthly* **107**(5) (2000) p. 462.
- [2] P. A. Damianou, Monic polynomials in $Z[x]$ with roots in the unit disc, *Amer. Math. Monthly*, 108 (2001) 253-257.
- [3] P. A. Damianou, On prime values of cyclotomic polynomials, arXiv:1101.1152 (2011).
- [4] Y. Gallot, Cyclotomic polynomials and Prime Numbers, <http://perso.orange.fr/yves.gallot/papers/cyclotomic.pdf>
- [5] Louis Leroux, Computing the torsion points of a variety defined by lacunary polynomials arXiv:0911.2594. (2009).
- [6] S. W. Golomb, Cyclotomic polynomials and factorization theorems, *Amer. Math. Monthly*, 85 (1978), 734-737.
- [7] J. Nieto On the divisibility of polynomials with integer coefficients. *Divulg. Mat.*, 11 (2003), 149-152.

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